SHORT-PULSE RADAR VIA ELECTROMAGNETIC WAVELETS *

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INTRODUCTION

The new theory of physical wavelets makes it possible to perform radar and sonar analysis directly in the space-time domain, based on fundamental principles underlying the emission, reflection, and reception of electromagnetic and acoustic waves[1,2]. Being independent of the Fourier transform and even of the usual (affine) wavelet transform, this formalism is therefore equally well suited for ultrawideband or short-pulse radar as for narrowband or continuous-wave radar. However, Fourier analysis does have a natural place in this theory and can be used easily when spectral questions are of interest.

A transmitting antenna following an arbitrary (possibly accelerating or nonlinear) space-time trajectory $\alpha(t)$ emits a physical (acoustic or electromagnetic) wavelet which is propagated in space by the appropriate Green function. This defines an *emission operator* E_α which, acting on any time signal $\psi(t)$, gives the emitted space-time wave $(E_\alpha \psi)(\mathbf{r}, t)$. The *reception operator* R_α is dual to E_α, measuring any incident space-time wave $F(\mathbf{r}, t)$ along the given antenna trajectory $\alpha(t)$ to produce the received time signal $(R_\alpha F)(t)$. Reflection is modeled as reception followed by re-emission, i.e., by the operator $E_\alpha R_\alpha$ transforming any incident space-time wave to the reflected space-time wave. Let the receiving antenna follow another arbitrary space-time trajectory $\gamma(t)$ (possibly different from the trajectory $\alpha(t)$ of the transmitter), let the target follow a third arbitrary space-time trajectory $\beta_T(t)$, and let the transmitted and received signals be $\psi(t)$ and $\chi(t)$, respectively. The objective is to estimate the target trajectory $\beta_T(t)$ from a knowledge of $\alpha(t), \gamma(t), \psi(t)$ and $\chi(t)$. This is achieved by maximizing the modulus of the the *normalized ambiguity functional* $\tilde{\chi}_N(\beta)$, obtained by matching the *actual* return $\chi(t)$ with the *computed* return due to a *trial trajectory* $\beta(t)$. When the radar is monostatic and the target is assumed to move uniformly in the radial direction, then $\tilde{\chi}_N(\beta)$ reduces to the usual *wideband ambiguity function*, which is just the ordinary *time-scale (wavelet) transform* of $\chi(t)$ with $\psi(t)$ as the basic wavelet. In the narrowband approximation, it reduces further to the usual time-frequency ambiguity function, which is a windowed Fourier transform of the *video signal* of the return. This shows that our "physical wavelet analysis" is a generalization of the usual ("mathematical") wavelet analysis, which is in turn a generalization of time-frequency analysis. In particular, our analysis applies equally well to *bistatic wideband*

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radar, where the Doppler effect can no longer be represented simply by scaling and hence the usual affine wavelet analysis breaks down.

In Reference 2, the transmitting and receiving antennas and the target were all assumed to be *points*, so that the trajectories $\alpha(t), \beta(t)$, and $\gamma(t)$ fully describe their motions. Consequenctly, the emitted and reflected waves are omnidirectional and hence are not very useful in practice. In this paper we generalize the above model to include *extended* antennas and targets, follwing an idea introduced by Heyman and Felsen[3]. The associated radar beams are much more useful since they have a measure of directivity.

EXTENDED PHYSICAL WAVELETS AND AMBIGUITY FUNCTIONALS

Suppose we are given an antenna located at the space point $\mathbf{x}$, emitting the response to an impulse at time t. Ignoring polarization for simplicity, the resulting wave at the observation point $\mathbf{x}'$ at time t' can be represented by a solution of the scalar wave equation, which we write as

$$K(\mathbf{x}', t' \,|\, \mathbf{x}, t),$$

with a source distribution appropriate to the antenna. If the antenna is very small (essentially a point) and omnidirectional, then $K(\mathbf{x}', t' \,|\, \mathbf{x}, t)$ is well approximated by the retarded Green function

$$G(\mathbf{x}' - \mathbf{x}, t' - t) = \frac{\delta(t' - t - |\mathbf{x}' - \mathbf{x}|/c)}{4\pi|\mathbf{x}' - \mathbf{x}|},$$

where c is the speed of light. Following Heyman and Felsen[3], a simple model can be formulated for an *extended* antenna by allowing *complex* antenna space-time coordinates $\mathbf{x} \to \mathbf{z} = \mathbf{x} + i\mathbf{y}$ and $t \to u = t + is$ and taking $K(\mathbf{x}', t' \,|\, \mathbf{z}, u)$ to be an analytic extension of the above retarded Green function.* Heyman and Felsen showed that for $|\mathbf{y}| < cs$, $K(\mathbf{x}', t' \,|\, \mathbf{z}, u)$ can be interpreted as a *pulsed beam field* emitted by a circular disk of radius $|\mathbf{y}|$ in the direction of $\mathbf{y}$. Thus $\mathbf{y}$ is a convenient "handle" by which the radius and orientation of the antenna can be controlled, without having to construct a messy model for the antenna involving a continuous distribution of point sources. More complicated extended antennas and arrays can be modeled by a distribution (continuous or discrete) of such complex source points.

To keep the notation uncluttered, we combine the space and time coordinates into a single symbol:

$$x' \equiv (\mathbf{x}', t') \in \mathbf{R}^4, \qquad z \equiv (\mathbf{z}, u) = x + iy \in \mathbf{C}^4,$$

where

$$x = (\mathbf{x}, t) \quad \text{and} \quad y = (\mathbf{y}, s).$$

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* Note that our convention differs from that of Reference 3 in that our positive-frequency time-harmonic waves vary as $e^{i\omega t}$ rather than $e^{-i\omega t}$. For this reason, the analytic continuation of the retarded Green function (using the analytic signal of the delta function) is to the upper-half time plane ($s > 0$) rather than the lower-half time plane. This is consistent with the convention used in Reference 1.

The condition $|\mathbf{y}| < cs$ means that the imaginary space-time four-vector y belongs to the *future cone*, so that z actually belongs to the complex *future tube*[1,4]

$$z \in \mathcal{T}_+ \equiv \{x + iy \in \mathbf{C}^4 : x \in \mathbf{R}^4 \text{ and } y = (\mathbf{y}, s) \text{ with } |\mathbf{y}| < cs\},$$

which is a four-dimensional generalization of the upper-half complex time plane.

Suppose now that our antenna executes an arbitrary motion, including possible rotations and accelerations. Using the complex source coordinates, this can be parameterized as

$$z = \alpha(t) = x(t) + iy(t), \quad \text{where } x(t) = (\mathbf{x}(t), t) \in \mathbf{R}^4 \text{ and } y(t) = (\mathbf{y}(t), s). \tag{1}$$

It is reasonable (but mathematically unnecessary) to assume that the radius of the antenna remains constant during the motion, so that $|\mathbf{y}(t)| = |\mathbf{y}(0)| \equiv R < cs$, although the direction of $\mathbf{y}(t)$ may vary to allow tracking, scanning, etc. While executing this motion, the antenna is fed an input time signal $\psi(t)$. Then the output beam is

$$\Psi_\alpha(x') = \int_{-\infty}^{\infty} dt \, K(x' \,|\, \alpha(t)) \, \psi(t).$$

For reasons explained in Reference 2, we call $\Psi_\alpha(x')$ the *extended physical wavelet* generated by $\psi(t)$ along the antenna motion $\alpha(t)$. Given $\alpha(t)$, we define the *emission operator* E_α as the operator transforming the time signal $\psi(t)$ to the space-time wave $\Psi_\alpha(x')$, i.e.,

$$(E_\alpha \psi)(x') \equiv \int_{-\infty}^{\infty} dt \, K(x' \,|\, \alpha(t)) \, \psi(t). \tag{2}$$

Thus E_α takes a function of one variable (the input signal) to a function of four variables (the output beam). On the other hand, if the antenna is used as a receiver, it converts space-time waves into time signals. Again, assume that the complex antenna motion $\alpha(t)$ is given as in (1). Then the simplest model for the received signal due to an incident wave $F(x')$ is

$$(R_\alpha F)(t) = g_\alpha F(\alpha(t)), \tag{3}$$

where g_α is a "gain factor." Thus R_α simply measures the field along the complex trajectory $\alpha(t)$. More complicated receivers can be formulated which measure derivatives of F along $\alpha(t)$. (In the full electromagnetic formalism, for example, R_α could measure the induced current rather than the field.) Since $\alpha(t)$ is complex, the "evaluation" of the field $F(x')$ at $x' = \alpha(t)$ must be defined in (3). For this we use the *analytic-signal transform* of F, which extends F to complex space-time[1,4]:

$$F(x + iy) \equiv \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} \, F(x + \tau y).$$

When $y = (\mathbf{0}, s)$ with $s > 0$, $F(x+iy)$ reduces to the usual Gabor analytic signal $F(\mathbf{x}, t+is)$ corresponding to $F(\mathbf{x}, t)$, with $\mathbf{x}$ regarded as an external parameter; this function is analytic in the upper-half complex time plane. It is further shown in Reference 1 that if $F(x')$ is any solution of the homogeneous wave equation (or Klein-Gordon equation[4]), then $F(x + iy)$ is analytic in the future tube $\mathcal{T}_+$ (i.e., $|\mathbf{y}| < cs$). The reception operator then evaluates the analytic-signal transform $F(x + iy)$ in its region of analyticity.

With emission and reception modeled by (2) and (3), we are almost ready to formulate a general radar problem. The only missing element is a model for *reflection*. In the spirit of regarding a scattered electromagnetic wave as being emitted by the current induced on the scatterer by the incident wave, we propose the following model: Suppose we are given an oriented circular "target" disk executing a motion described by a complex space-time trajectory $\alpha(t) = x(t) + iy(t)$ as in (1). Again, we interpret the imaginary position vector $\mathbf{y}(t)$ as defining the radius and orientation of the disk. (To say that the disk is "oriented" means that its two sides are not equivalent; for example, one side could be reflective while the other side is not. Then every unit vector $\hat{\mathbf{y}} = \mathbf{y}/|\mathbf{y}|$ corresponds to a unique orientation of the disk. This is useful if, for example, we approximate a complicated target by patching together disks of various sizes and orientations, as in Section 10.2 of Reference 1; their non-reflecting sides should then be oriented towards the interior.) A given space-time wave $F(x')$ will now be assumed to be reflected from the disk as follows: First the disk acts as a *receiver*, then as a *transmitter*. Thus the reflected wave is

$$F_{\text{refl}}(x') = (E_\alpha R_\alpha F)(x') = g_\alpha \int_{-\infty}^{\infty} dt\, K(x'\,|\,\alpha(t))\, F(\alpha(t)).$$

Note that in the present context, the original "gain factor" g_α is re-interpreted as a *reflection coefficient*. When a complicated target is patched together from circular targets of various radii and orientations, the reflection coefficient becomes a function defined over the target surface as desired.

The ambiguity functional formalism developed in Reference 2 generalizes easily and naturally to the present setting of extended physical wavelets. Given the outgoing time signal ψ and the motions α, β, and γ of the transmitter, target, and receiver (all complex), our model for the time signal received at γ is

$$
\begin{aligned}
\psi_\beta(t'') &= (R_\gamma E_\beta R_\beta E_\alpha \psi)(t'') \\
&= g_\gamma g_\beta \iint dt'\, dt\, K(\gamma(t'')\,|\,\beta(t'))\, K(\beta(t')\,|\,\alpha(t))\, \psi(t).
\end{aligned}
\tag{4}
$$

Of course, the received signal depends functionally on all three trajectories α, β, γ, as is evident from the right-hand side of (4). But to simplify the notation, we have suppressed the dependence on the *known* trajectories α and γ and displayed only the dependence on the target trajectory β. To estimate the *actual* target trajectory $\beta_T(t)$, we compute $\psi_\beta(t)$ for a *trial trajectory* $\beta(t)$ and match the result with the actual return $\chi(t)$ by taking the inner product of the two time signals. We denote the result by $\tilde{\chi}(\beta)$, which we call the *ambiguity functional* of the return:

$$
\begin{aligned}
\tilde{\chi}(\beta) &\equiv \langle \chi, \psi_\beta \rangle \equiv \int_{-\infty}^{\infty} dt''\, \chi(t'')\, \psi_\beta(t'') \\
&= \iiint dt''\, dt'\, dt\, \chi(t'')\, K(\gamma(t'')\,|\,\beta(t'))\, K(\beta(t')\,|\,\alpha(t))\, \psi(t).
\end{aligned}
\tag{5}
$$

(We assume that $\psi(t)$ and $\chi(t)$ are real; if they are complex, then $\chi(t)$ should be replaced by its complex conjugate in (5).) Assuming that $\psi_\beta(t)$ and $\chi(t)$ have finite energies $\|\psi_\beta\|^2$ and $\|\chi\|^2$, the Schwarz inequality implies that

$$
\begin{aligned}
|\tilde{\chi}(\beta)| &= |\langle \chi, \psi_\beta \rangle| \leq \|\chi\|\,\|\psi_\beta\| \\
|\tilde{\chi}(\beta)| &= \|\chi\|\,\|\psi_\beta\| \iff \chi(t) = C\psi_\beta(t).
\end{aligned}
\tag{6}
$$

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Therefore, to estimate the true target trajectory $\beta_T(t)$, we need to maximize the *normalized ambiguity functional*

$$\tilde{\chi}_N(\beta) \equiv \frac{\tilde{\chi}(\beta)}{\|\psi_\beta\|} \, .$$

By (6),

$$|\tilde{\chi}_N(\beta)| \le \|\chi\| \quad \text{and} \quad |\tilde{\chi}_N(\beta)| = \|\chi\| \iff \chi(t) = C\psi_\beta(t).$$

Equivalently, we can minimize the *error functional* defined by

$$\mathcal{E}(\beta) \equiv 1 - \frac{|\chi(\beta)|}{\|\chi\| \, \|\psi_\beta\|} \, ,$$

since the Schwarz inequality states that

$$0 \le \mathcal{E}(\beta) \le 1 \quad \text{and} \quad \mathcal{E}(\beta) = 0 \iff \chi(t) = C\psi_\beta(t).$$

Thus $|\tilde{\chi}_N(\beta)|$ and $\mathcal{E}(\beta)$ attain their maximum and minimum values, respectively, only when the trial return is indistinguishable from the actual return. Of course, this does not guarantee that the trial trajectory $\beta(t)$ coincides with the actual target trajectory $\beta_T(t)$, since the return does not, in general, uniquely determine the target trajectory. That is, *the functionals $\tilde{\chi}_N(\beta)$ and $\mathcal{E}(\beta)$ are generally not one-to-one.* The class of all trajectories β such that $\tilde{\chi}_N(\beta) = \tilde{\chi}_N(\beta_T)$ or, equivalently, $\mathcal{E}(\beta) = \mathcal{E}(\beta_T)$, represents the inherent *ambiguity* of the radar problem. A problem of obvious importance is to find outgoing signals $\psi(t)$ which minimize this ambiguity class.

We have assumed above that the return is due to a reflection from a *single* target. If N distinct targets are involved, then we can approximate the return as a superposition

$$\psi_{\beta_1,\beta_2,\dots,\beta_N} \approx \psi_{\beta_1} + \cdots + \psi_{\beta_N} \, . \tag{7}$$

As noted, (7) is an approximation because it ignores *multiple reflections.* Although these can often be ignored, they can also cause resonances (ringing), hence must sometimes be taken into account. This can be easily done, in principle. For example, the signal received by the doubly-reflecting path $\alpha \to \beta_m \to \beta_n \to \gamma$ is

$$\psi_{\beta_m,\beta_n} = R_\gamma E_{\beta_n} R_{\beta_n} E_{\beta_m} R_{\beta_m} E_\alpha \psi \, ,$$

which can be immediately converted to a triple integral by using the definitions (2) and (3). Sums of contributions from various "trial" scattering paths may then be matched with the actual return, defining a generalized ambiguity functional

$$\tilde{\chi}(\beta_1, \beta_2, \dots, \beta_N) \equiv \langle \chi, \psi_{\beta_1,\beta_2,\dots,\beta_N} \rangle,$$

and the Schwarz inequality may be used as in the case of a single path to optimize the match.

This method is reminiscent of *Feynman diagrams*[5], where fundamental processes are represented by multiple integrals with corresponding intuitive diagrams. Because the physics is built into the formalism from the beginning through the Green functions, our

model can handle such complications in a conceptually straightforward (if computationally nontrivial) way. The resemblance to Feynman diagrams is no coincidence, and the present formalism may be modified to include quantum (photonic) aspects of radar simply by using *Feynman propagators* in place of the retarded Green functions.

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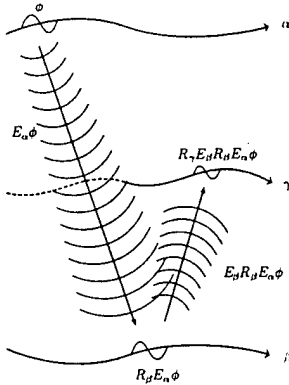
Remote Sensing via Physical Wavelets [*]

A short course by

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PLAN: Four 90-minute lectures

1. Generalized Radar Analysis with Point Wavelets
2. Reduction to Time-Scale Analysis
3. Reduction to Time-Frequency Analysis
4. Directivity: Radar with Pulsed-Beam Wavelets

No previous knowledge of wavelet analysis or time-frequency analysis will be assumed. Both will be seen to emerge naturally from generalized radar analysis!

COURSE SUMMARY

Problem: Estimate *arbitrary target motions* given *arbitrary, independent* motions of transmitting and receiving antennas. **Method:** Match *measured return* with *trial return* based on trial target motions. This gives the *ambiguity functional*, maximized when the two returns coincide. For *monostatic* radar with *uniform target motion*, it reduces to the *wideband ambiguity function* (time-scale transform) of the return. For narrowband signals, it further reduces to the usual time-frequency transform. To obtain *directivity*, use *pulsed beams*. **Time-frequency analysis is an approximation to wavelet analysis, which is an approximation to "physical" wavelet analysis.**

LECTURE 1
Physical Wavelets and Ambiguity Functionals

1. Emission, Reception and Reflection

For simplicity, we ignore polarization. Since our motivation is radar and remote sensing, we use the language of "antennas" even though we are really doing acoustics here. The concepts easily extend to EM.

> *Space-time notation:* $(\mathbf{x}, t) \equiv x \in \mathbf{R}^4$

Time-dependent source distribution: $J(\mathbf{x}, t) \equiv J(x)$

The resulting space-time wave $F(x) \equiv F(\mathbf{x}, t)$ satisfies the *wave equation*

$$\left(\partial_{ct}^2 - \nabla_{\mathbf{x}}^2 \right) F(x) = J(x)$$

in free space. The *causal* solution is given by

$$F(x) = \int_{\mathbf{R}^4} G(x - x') J(x') \, d^4 x',$$

where G is the *retarded Green function*

$$G(x) \equiv G(\mathbf{x}, t) = \frac{\delta(t - |\mathbf{x}|/c)}{4\pi |\mathbf{x}|} .$$

Now consider a *point source* moving along an arbitrary trajectory

$$\mathbf{x} = \mathbf{r}(\tau).$$

For simplicity denote the corresponding trajectory in *space-time* by α:

$$\alpha(\tau) = (\mathbf{r}(\tau), \tau) \in \mathbf{R}^4.$$

While moving, the source emits a time signal $\psi(\tau)$. This moving point transmitter is described by a source function concentrated on $\alpha(\tau)$, namely

$$J(x) = \int_{-\infty}^{\infty} d\tau \, \delta^{(4)}(x - \alpha(\tau)) \psi(\tau).$$

The emitted wave is then

$$\Psi_\alpha(x) \equiv \int_{-\infty}^{\infty} d\tau \, G(x - \alpha(\tau)) \psi(\tau)$$

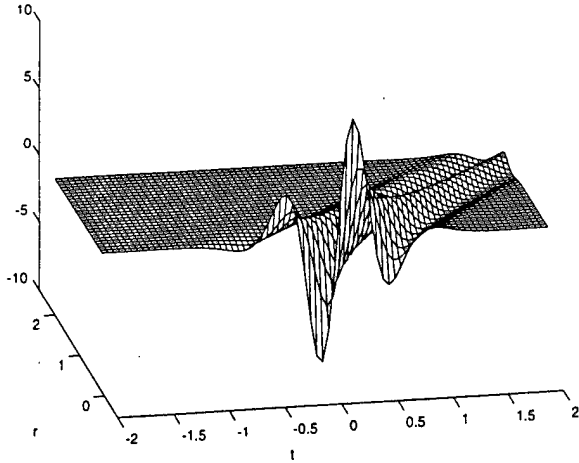$$= \int_{-\infty}^{\infty} d\tau \, G(\mathbf{x} - \mathbf{r}(\tau), t - \tau) \psi(\tau).$$

For example, if $\mathbf{r}(t) \equiv \mathbf{0}$, then $\Psi_\alpha(x)$ is the *spherical wave*

$$\Psi_\alpha(\mathbf{x}, t) = \frac{\psi(t - |\mathbf{x}|/c)}{4\pi|\mathbf{x}|} = \frac{\psi(t - r/c)}{4\pi r} \equiv \Psi(r, t),$$

as shown below.

A physical wavelet $\Psi(r, t)$ with stationary source

1-3

If $\mathbf{r}(\tau) = \mathbf{r}_0 + \mathbf{v}\tau$, then $\Psi_\alpha(x)$ *is a translated (delayed) and Doppler scaled version of* ψ [1]. However, the amount of scaling depends on the direction of the observer relative to the velocity $\mathbf{v}$ of the source, as is intuitively and geometrically clear.

Translations and scalings are the "raw material" of wavelets, hence for *general* $\alpha(\tau)$ we call $\Psi_\alpha(x)$ the **acoustic wavelet** *generated by* $\psi(\tau)$ *along* $\alpha(\tau)$ [1–3].

EM wavelets are defined similarly as waves generated by moving elementary currents.

Here "elementary" means *point sources*, but this can be generalized to *extended* sources [4].

Given a trajectory α, define the **emission operator** by $(E_\alpha \psi)(x) = \Psi_\alpha(x)$, i.e.,

$$(E_\alpha \psi)(x) = \int_{-\infty}^{\infty} d\tau \; G(x - \alpha(\tau))\psi(\tau). \qquad (1)$$

E_α converts any given *time signal* $\psi(\tau)$ to the emitted *space-time wave* $\Psi_\alpha(x)$ (*one* to *four* variables).

1-4

Reception works the other way: A receiver moving along $\alpha(\tau)$ converts a space-time wave $F(x)$ into a time signal by *measuring* the wave along $\alpha(\tau)$. This defines the **reception operator**

$$(R_\alpha F)(\tau) = g_\alpha F(\alpha(\tau)) \qquad (2)$$

where g_α is a gain factor. Thus R_α converts functions of *four* variables to functions of *one* variable.

More general reception operators are possible. For example, in EM, we can measure the induced *current*, which involves *derivatives* of the field. The above model is the simplest.

Now consider a *point target* moving along an arbitrary trajectory $\beta(\tau)$ through a wave $F(x)$.

We model the *reflected wave* $F_{\text{refl}}(x)$ by assuming that *the target acts first as a receiver, then as a trans-*

1-5

mitter:

$$F_{\text{refl}}(x) = (E_\beta R_\beta F)(x)$$
$$= g_\beta \int_{-\infty}^{\infty} d\tau \; G(x - \beta(\tau))F(\beta(\tau)). \qquad (3)$$

This is similar to the EM case, where the incident wave induces a current on the target, which in turn radiates the scattered wave.

Problem: Given a transmitter following an arbitrary trajectory $\alpha(\tau)$ and a receiver following an arbitrary trajectory $\gamma(\tau)$. Given the transmitted signal $\psi(\tau)$ and the received signal $\chi(\tau)$. Assuming there is a single target, estimate its trajectory $\beta_T(\tau)$.

2. The Ambiguity Functional

Let $\beta(\tau)$ be a *trial target trajectory*. Then emission of ψ from α, reflection from β, and reception at γ are given by

$$\psi \to E_\alpha \psi \to E_\beta R_\beta E_\alpha \psi \to R_\gamma E_\beta R_\beta E_\alpha \psi.$$
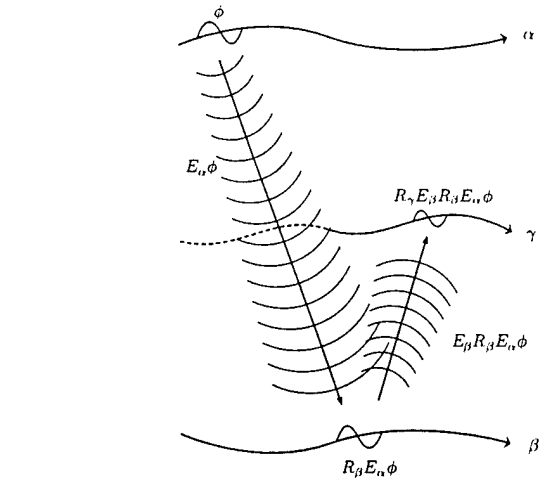
1-6

Hence the *trial return* at γ is

$$\psi_\beta \equiv R_\gamma E_\beta R_\beta E_\alpha \psi,$$

where only the dependence on the trial trajectory β has been made explicit. Of course, ψ_β also depends *implicitly* on the *known* trajectories α and γ.

Inserting the definitions (1) and (2) gives

$$\psi_\beta(\tau) = g_\gamma g_\beta \iint d\tau' \, d\tau' \, G(\gamma(\tau) - \beta(\tau'))$$
$$\times \, G(\beta(\tau') - \alpha(\tau'')) \, \psi(\tau'').$$

This is the *computed trial return* assuming a trial trajectory β.

ϕ is emitted by α, reflected by β, received by γ.

To estimate β, **match** ψ_β with the *actual* return χ:

$$\langle \chi, \psi_\beta \rangle \equiv \int_{-\infty}^{\infty} d\tau \, \chi(\tau) \, \psi_\beta(\tau),$$

where we assumed that $\psi(\tau)$ is *real*, hence also $\psi_\beta(\tau)$ and $\chi(\tau)$. The right-hand side depends on the trajectory β *as a whole*, hence it is a *functional* of β. We

call it the **ambiguity functional** and denote it by $\tilde\chi(\beta)$. Explicitly,

$$\tilde\chi(\beta) = g_\beta g_\gamma \iiint d\tau \, d\tau' \, d\tau'' \, \chi(\tau) \, G(\gamma(\tau) - \beta(\tau'))$$
$$\times \, G(\beta(\tau') - \alpha(\tau'')) \, \psi(\tau'').$$

In practice, both $\psi_\beta(\tau)$ and $\chi(\tau)$ have *finite energy*:

$$\|\psi_\beta\|^2 \equiv \int_{-\infty}^{\infty} d\tau \, \psi_\beta(\tau)^2 < \infty$$

$$\|\chi\|^2 \equiv \int_{-\infty}^{\infty} d\tau \, \chi(\tau)^2 < \infty.$$

Then $\tilde\chi(\beta) = \langle \chi, \psi_\beta \rangle$ satisfies *Schwarz's inequality*:

$$|\langle \chi, \psi_\beta \rangle| \le \|\chi\| \, \|\psi_\beta\| \text{ for all } \beta, \text{ and}$$
$$|\langle \chi, \psi_\beta \rangle| = \|\chi\| \, \|\psi_\beta\| \iff \chi(\tau) = C \psi_\beta(\tau)$$

for some constant C.

Thus *to estimate the actual target trajectory β_T, vary the trial trajectory β to maximize the **normalized** ambiguity functional*

$$\tilde\chi_N(\beta) \equiv \frac{\tilde\chi(\beta)}{\|\psi_\beta\|}.$$

Then

$$\left| \tilde\chi_N(\beta) \right| \le \|\chi\|$$
$$\left| \tilde\chi_N(\beta) \right| = \|\chi\| \iff \chi(\tau) = C \psi_\beta(\tau).$$

We will show that $\tilde\chi_N(\beta)$ is a generalization of the usual wideband and narrowband radar ambiguity functions.

Even better: Define the **error functional**

$$\mathcal{E}(\beta) \equiv 1 - \frac{|\langle \chi, \psi_\beta \rangle|^2}{\|\chi\|^2 \, \|\psi_\beta\|^2} = 1 - \frac{\left| \tilde\chi_N(\beta) \right|^2}{\|\chi\|^2}.$$

Then Schwarz's inequality reads

$$0 \le \mathcal{E}(\beta) \le 1 \text{ for all } \beta, \text{ and}$$
$$\mathcal{E}(\beta) = 0 \iff \chi(\tau) = C \psi_\beta(\tau).$$

> To estimate the target trajectory, vary the trial trajectory β to minimize $\mathcal{E}(\beta)$. For a given β, we may regard the *value* of $\mathcal{E}(\beta)$ as an estimate of the error.

Warning: Even $\mathcal{E}(\beta) = 0$ does not guarantee that β is the true target trajectory, only that the trial return ψ_β is indistinguishable from the actual return χ. But clearly we cannot do any better than that! For example, in monostatic radar with point sources, all target trajectories with the same $|\mathbf{r}(t)|$ have the same $\mathcal{E}(\beta)$.

It is impractical to try *all possible* trajectories β. Any prior knowledge or reasonable guess can be used to restrict the class. For example, assume the target moves with *constant acceleration* $\mathbf{a}$:

$$\mathbf{r}(\tau) = \mathbf{r}_0 + \mathbf{v}_0\tau + \frac{1}{2}\,\mathbf{a}\tau^2.$$

Then $\mathcal{E}(\beta)$ reduces to an ordinary *function* of the target parameters:

$$\mathcal{E}(\beta) \rightarrow \mathcal{E}(\mathbf{r}_0, \mathbf{v}_0, \mathbf{a}),$$

which must be minimized to get the best parameters in this class.

3. Monostatic Radar, Uniform Target Motion

To see the connection with ambiguity functions, assume

♠ *Monostatic Radar:* The transmitter and receiver are both at rest at the origin:

$$\alpha(t) = \gamma(t) = (\mathbf{0}, t).$$

♣ *Uniform, radial motion:* The target motion is

$$\mathbf{r}(t) = r(t)\mathbf{n}, \quad \text{where} \quad r(t) \equiv |\mathbf{r}(t)| = r_0 + vt.$$

Thus

$$\beta(t) = ((r_0 + vt)\mathbf{n}, t)$$

is parameterized by two just scalars: The **range** $r_0 > 0$ and the **range rate** v. If $v < 0$, the target is *closing in*, and if $v > 0$, it is *receeding*.

Theorem: *Under the assumptions* ♠ *and* ♣, *the trial return is given by*

$$\psi_\beta(t) = A(t)\,\psi(\sigma t - \tau), \tag{4}$$

where

$$A(t) = \frac{g_\gamma g_\beta(1 + v/c)}{16\pi^2 r(t)^2} = \text{attenuation factor}$$

$$\sigma = \frac{c - v}{c + v} = \text{time scaling factor} \tag{5}$$

$$\tau = \frac{2r_0}{c + v} = \text{time shift}.$$

Each quantity in (5) has a simple interpretation:

1. The attenuation factor $A(t)$ implies that the *instantaneous power* of $\psi_\beta(t)$ is

$$\psi_\beta(t)^2 \propto r(t)^{-4},$$

in agreement with the *radar range equation*.

2. The scale factor σ represents the Doppler effect:

$$-c < v < 0 \;\Rightarrow\; \sigma > 1 \text{ and } \psi \text{ is \textbf{compressed}.}$$
$$0 < v < c \;\Rightarrow\; 0 < \sigma < 1 \text{ and } \psi \text{ is \textbf{stretched}.}$$

3. The shift $\tau > 0$ represents the *time delay* due to the finite propagation speed.

Proof: Denote the wavelet received at $\beta(t)$ (before re-emission) by $\psi'(t)$. Then

$$\begin{aligned}
\psi'(t) &= (R_\beta E_\alpha \psi)(t) \\
&= g_\beta \int dt'\, G(\beta(t) - \alpha(t'))\,\psi(t') \\
&= g_\beta \int dt'\, G(r(t)\mathbf{n}, t - t')\,\psi(t') \\
&= g_\beta \int dt'\, \frac{\delta(t - t' - r(t)/c)}{4\pi r(t)}\,\psi(t') \\
&= g_\beta\, \frac{\psi(t - r(t)/c)}{4\pi r(t)}.
\end{aligned}$$

The reflected wavelet received at $\gamma(t) = (\mathbf{0}, t)$ is therefore

$$\psi_\beta(t) = g_\gamma \int dt' \, G(\gamma(t) - \beta(t')) \, \psi'(t')$$

$$= g_\gamma \int dt' \, G(-r(t')\mathbf{n}, t - t') \, \psi'(t') \qquad (6)$$

$$= g_\gamma \int dt' \, \frac{\delta(t - t' - r(t')/c)}{4\pi r(t')} \, \psi'(t').$$

But

$$t - t' - \frac{r(t')}{c} = \frac{c(t - t') - r_0 - vt'}{c}$$

$$= \frac{ct - r_0 - (c + v)t'}{c}$$

$$= -\frac{c + v}{c} \left[t' - \frac{ct - r_0}{c + v} \right]$$

$$= -\frac{c + v}{c} \, (t' - T),$$

where

$$T \equiv \frac{ct - r_0}{c + v}.$$

Thus

$$\delta(t - t' - r(t')/c) = \delta\left(\frac{c + v}{c} \, (t' - T) \right)$$

$$= \frac{c}{c + v} \, \delta(t' - T).$$

Therefore (6) gives

$$\psi_\beta(t) = g_\gamma \frac{c}{c + v} \, \frac{\psi'(T)}{4\pi r(T)}$$

$$= g_\gamma g_\beta \frac{c}{c + v} \, \frac{\psi(T - r(T)/c)}{[4\pi r(T)]^2}.$$

But

$$r(T) = r_0 + v\left(\frac{ct - r_0}{c + v} \right) = \frac{c}{c + v} \, r(t),$$

hence

$$T - \frac{r(T)}{c} = T - \frac{r(t)}{c + v} = \frac{ct - r_0}{c + v} - \frac{r_0 + vt}{c + v}$$

$$= \left(\frac{c - v}{c + v} \right) t - \frac{2r_0}{c + v} \equiv \sigma t - \tau.$$

Putting all this together gives

$$\psi_\beta(t) = \frac{g_\gamma g_\beta (1 + v/c)}{16\pi^2 r(t)^2} \, \psi(\sigma t - \tau)$$

$$= A(t) \, \psi(\sigma t - \tau). \qquad \text{QED}$$

Under ♠ and ♣, the ambiguity functional reduces to

$$\bar\chi(\beta) = \int_{-\infty}^{\infty} dt \, A(t) \, \dot\psi(\sigma t - \tau) \, \chi(t) \equiv \tilde\chi(\sigma, \tau).$$

Below we will see that this is essentially the **wavelet transform** of $\chi(t)$, also known as the **wideband amiguity function**.

LECTURE 2
Reduction to Time-Scale Analysis

Recall that for monostatic radar (♠) and uniform, radial target motion (♣), the trial return is

$$\psi_\beta(t) = \frac{g_\gamma g_\beta (1 + v/c)}{16\pi^2 r(t)^2} \, \psi(\sigma t - \tau) \equiv A(t) \, \psi(\sigma t - \tau).$$

In practice,

$$r(t) \approx r(0) = r_0$$

during the reflection time interval. Hence

$$A(t) \approx \frac{g_\gamma g_\beta (1 + v/c)}{16\pi^2 r_0^2} \equiv A_0$$

and the trial return is

$$\psi_\beta(t) = A_0 \, \psi(\sigma t - \tau).$$

To connect σ with the Doppler effect, suppose the outgoing signal is *time-harmonic* (CW):

$$\psi(t) = e^{2\pi i f t}, \quad f > 0.$$

Then the trial return is

$$\psi_\beta(t) = A_0\, e^{2\pi i f(\sigma t - \tau)} = A_0\, e^{-2\pi i f \tau}\, e^{2\pi i \sigma f t}.$$

The factor $e^{-2\pi i f \tau}$ represents the time delay, and $e^{2\pi i \sigma f t}$ is the reflected wave with the *Doppler-scaled* frequency σf. As expected,

$$-c < v < 0 \;\Rightarrow\; \sigma > 1 \;\Rightarrow\; \sigma f > f$$
$$0 < v < c \;\Rightarrow\; 0 < \sigma < 1 \;\Rightarrow\; \sigma f < f.$$

Only in the narrowband approximation can $f \to \sigma f$ be replaced by a *uniform frequency shift* $f \to f + \phi$.

2-2

Returning to the general signal $\psi(t)$, we have

$$\|\psi_\beta\|^2 = A_0^2 \int_{-\infty}^{\infty} dt\; \psi(\sigma t - \tau)^2 = \frac{A_0^2}{\sigma}\,\|\psi\|^2.$$

We may assume that the outgoing signal $\psi(t)$ has unit energy, i.e.,

$$\|\psi\|^2 \equiv \int_{-\infty}^{\infty} dt\; \psi(t)^2 = 1, \;\Rightarrow\; \|\psi_\beta\|^2 = \frac{A_0^2}{\sigma}\;.$$

Then the normalized ambiguity functional reduces to

$$\tilde\chi_N(\beta) \equiv \frac{\langle \psi_\beta, \chi \rangle}{\|\psi_\beta\|}$$
$$= \sqrt{\sigma} \int_{-\infty}^{\infty} dt\; \psi(\sigma t - \tau)\chi(t)$$
$$\equiv \langle \psi_{\sigma,\tau}, \chi \rangle,$$

where

$$\psi_{\sigma,\tau}(t) \equiv \sqrt{\sigma}\,\psi(\sigma t - \tau)$$

is a scaled and translated version of $\psi(t)$ with unit energy. Note that $\|\psi_{\sigma,\tau}\| = \|\psi\| = 1$.

2-3

The *set* of functions

$$\{\psi_{\sigma,\tau} : \sigma > 0 \text{ and } \tau = \text{ any real number }\}$$

is called the **wavelet family** generated by ψ, and ψ is called the **basic** (or "mother") **wavelet**.

The normalized ambiguity functional thus reduces to an *ordinary function* of the target parameters (σ, τ),

$$\tilde\chi_N(\beta) \to \tilde\chi(\sigma,\tau) \equiv \langle \psi_{\sigma,\tau}, \chi \rangle$$
$$= \sqrt{\sigma} \int_{-\infty}^{\infty} dt\; \psi(\sigma t - \tau)\chi(t),$$

which is called the **wavelet transform** of $\chi(t)$ with respect to the wavelet family $\{\psi_{\sigma,\tau}\}$.

Note that for radar, only $\psi_{\sigma,\tau}$ with

$$\tau \equiv \frac{2r_0}{c+v} > 0 \quad \text{and} \quad \sigma \equiv \frac{c-v}{c+v} \approx 1$$

are needed. This will also be understood in the context of wavelet analysis. For now let τ be any real number and $\sigma > 0$.

2-4

Our assumptions ♠ and ♣ say nothing about the outgoing signal $\psi(t)$. In particular, $\psi(t)$ can be an **ultra-short pulse** or an **ultra-wideband** signal.

Thus in the radar literature, $\tilde\chi(\sigma, \tau)$ is known as the **wideband ambiguity function** of the return $\chi(t)$.

In fact, the parameter σ has a meaning even when $\psi(t)$ is an ultra-short pulse, since the "backscattering transformation"

$$\psi(t) \to \psi_{\sigma,\tau}(t) = \sqrt{\sigma}\,\psi(\sigma t - \tau)$$

occurs entirely in the *time domain*.

When $\psi(t)$ is *narrowband*, we will see that the *Doppler scale factor* σ can be replaced by a *Doppler frequency shift*. For this reason we call (σ, τ) the **wideband target parameters**.

The general method of ambiguity functionals and error functionals now takes the following reduced form.

The Schwarz inequality gives

$$\left| \tilde\chi(\sigma,\tau) \right| = \left| \langle \psi_{\sigma,\tau}, \chi \rangle \right| \le \|\psi_{\sigma,\tau}\|\,\|\chi\| = \|\chi\|,$$
$$\left| \tilde\chi(\sigma,\tau) \right| = \|\chi\| \iff \chi(t) = C\psi_{\sigma,\tau}(t).$$

2-5

To find the best wideband parameters, we must therefore *maximize* $|\tilde{\chi}(\sigma, \tau)|$.

> Under ♠ and ♣, our *physical wavelet analysis* of the return by its ambiguity functional reduces to ordinary ("mathematical") wavelet analysis by the time-scale wavelet transform!

Altenatively, we can define the **wideband error function**

$$\mathcal{E}(\sigma, \tau) \equiv 1 - \frac{|\langle \psi_{\sigma,\tau}, \chi \rangle|^2}{\|\psi_{\sigma,\tau}\|^2 \|\chi\|^2} = 1 - \frac{|\tilde{\chi}(\sigma, \tau)|^2}{\|\chi\|^2}.$$

Then the objective is to minimize $\mathcal{E}(\sigma, \tau)$, since

$$0 \leq \mathcal{E}(\sigma, \tau) \leq 1$$
$$\mathcal{E}(\sigma, \tau) = 0 \iff \chi(t) = C\psi_{\sigma,\tau}(t).$$

Furthermore, the *value* $\mathcal{E}(\sigma, \tau)$ gives an estimate of the error in the trial return $\psi_{\sigma,\tau}(t)$.

Once the wideband parameters (σ, τ) are estimated, the range and range rate are given by

$$r_0 = \frac{c\tau}{1+\sigma} \quad \text{and} \quad v = \left(\frac{1-\sigma}{1+\sigma}\right)c.$$

To understand the connection between time-scale analysis and time-frequency analysis, look at the wavelet transform in the frequency domain.

We use the Fourier transform notation

$$\hat{\chi}(f) \equiv \int_{-\infty}^{\infty} dt \ e^{-2\pi i f t} \chi(t)$$
$$\chi(t) = \int_{-\infty}^{\infty} df \ e^{2\pi i f t} \hat{\chi}(f).$$

Then

$$\hat{\psi}_{\sigma,\tau}(f) \equiv \int_{-\infty}^{\infty} dt \ e^{-2\pi i f t} \sqrt{\sigma} \psi(\sigma t - \tau)$$
$$= \frac{1}{\sqrt{\sigma}} e^{-2\pi i f \tau/\sigma} \hat{\psi}(f/\sigma). \tag{1}$$

Parseval's identity states that the inner product in the time domain equals the inner product in the frequency domain, hence (since $\hat{\psi}_{\sigma,\tau}(f)$ is *complex*)

$$\tilde{\chi}(\sigma, \tau) \equiv \langle \psi_{\sigma,\tau}, \chi \rangle = \langle \hat{\psi}_{\sigma,\tau}, \hat{\chi} \rangle$$
$$= \int_{-\infty}^{\infty} df \ \hat{\psi}_{\sigma,\tau}(f)^* \hat{\chi}(f).$$

Inserting (1) gives the **frequency domain version of the wavelet transform:**

> $$\tilde{\chi}(\sigma, \tau) = \frac{1}{\sqrt{\sigma}} \int_{-\infty}^{\infty} df \ e^{2\pi i f \tau/\sigma} \hat{\psi}(f/\sigma)^* \hat{\chi}(f).$$
♡

We require ψ to satisfy the **oscillation condition**

> $$\int_{-\infty}^{\infty} dt \ \psi(t) = 0, \quad \text{i.e.,} \quad \hat{\psi}(0) = 0. \tag{$\mathcal{O}$}$$

That is, $\psi(t)$ has no "DC component".

This is physically reasonable: *Since the DC component is constant (static), it cannot propagate and may as well be removed.* In wavelet analysis, a requirement closely related to ($\mathcal{O}$) is called the **admissibility condition.**

If $\psi(t)$ is integrable, then $\hat{\psi}(f)$ is continuous and

$$\hat{\psi}(f) \to 0 \quad \text{as} \quad f \to 0.$$

Since $\psi(t)$ has finite energy, we also have

$$\hat{\psi}(f) \to 0 \quad \text{as} \quad |f| \to \infty.$$

Suppose, in fact, that $\hat{\psi}(f)$ is a **bandpass filter**, i.e.,

$$\hat{\psi}(f) \approx 0 \quad \text{outside the double band} \quad f_1 \leq |f| \leq f_2$$

for some given frequencies $0 < f_1 < f_2$. Then

$$\frac{1}{\sqrt{\sigma}} \hat{\psi}(f/\sigma) \approx 0 \quad \text{outside of} \quad \sigma f_1 \leq |f| \leq \sigma f_2.$$

That is, $\sigma^{-1/2}\hat{\psi}(f/\sigma)$ is a **scaled bandpass filter.**

We can thus interpret $\heartsuit$ as follows:

1. *Localize* the spectrum $\hat{\chi}(f)$ using the scaled filter:

$$\hat{\chi}(f) \to \frac{1}{\sqrt{\sigma}} \hat{\psi}(f/\sigma)^* \hat{\chi}(f) \equiv \hat{\chi}_\sigma(f).$$

2. Take the *inverse Fourier transform* to get $\tilde{\chi}(\sigma, \tau)$:

$$\hat{\chi}_\sigma(f) \to \int_{-\infty}^{\infty} df \ e^{2\pi i f \tau / \sigma} \hat{\chi}_\sigma(f) = \tilde{\chi}(\sigma, \tau).$$

> The wavelet transform $\tilde{\chi}$ can be interpreted as a **scale-windowed inverse Fourier transform** (SWIFT) of $\hat{\chi}$.

In the narrowband approximation, this view will connect the time-scale analysis smoothly with the usual time-frequency analysis, where the window **slides** instead of being scaled; see also [3].

But what can wavelet analysis contribute to radar? The *wavelet inversion formula*, for one.

> • Given only $\tilde{\chi}(\sigma, \tau)$, find $\chi(t)$.

This can serve to model a *dense target environment*. To find $\chi(t)$ from $\tilde{\chi}(\sigma, \tau)$, note that $\heartsuit$ says

$$\tilde{\chi}(\sigma, \tau) = \mathcal{F}^{-1} \left\{ \frac{1}{\sqrt{\sigma}} \hat{\psi}(f/\sigma)^* \hat{\chi}(f) \right\} (\tau/\sigma).$$

Taking the inverse Fourier transform with respect to τ/σ gives

$$\sqrt{\sigma} \hat{\psi}(f/\sigma)^* \hat{\chi}(f) = \int_{-\infty}^{\infty} d\tau \ e^{-2\pi i f \tau / \sigma} \tilde{\chi}(\sigma, \tau).$$

Multiply both sides by $\sigma^{-1/2} \hat{\psi}(f/\sigma)$ and write

$$S_\psi(f) \equiv |\hat{\psi}(f)|^2$$

for the *spectral energy density* of ψ. Then

$$S_\psi(f/\sigma) \hat{\chi}(f) = \int_{-\infty}^{\infty} d\tau \ \frac{1}{\sqrt{\sigma}} \ e^{-2\pi i f \tau / \sigma} \hat{\psi}(f/\sigma) \tilde{\chi}(\sigma, \tau)$$

$$= \int_{-\infty}^{\infty} d\tau \ \hat{\psi}_{\sigma, \tau}(f) \tilde{\chi}(\sigma, \tau).$$

Now integrate over σ with an arbitrary *weight function* $w(\sigma)/\sigma$:

$$\int_0^{\infty} \frac{d\sigma}{\sigma} \ w(\sigma) S_\psi(f/\sigma) \hat{\chi}(f)$$
$$= \int_0^{\infty} \frac{d\sigma}{\sigma} \int_{-\infty}^{\infty} d\tau \ \hat{\psi}_{\sigma, \tau}(f) \tilde{\chi}(\sigma, \tau).$$

Define

$$W(f) \equiv \int_0^{\infty} \frac{d\sigma}{\sigma} \ w(\sigma) S_\psi(f/\sigma), \qquad (2)$$

the *frequency filter* corresponding to $w(\sigma)$. Then

$$W(f) \hat{\chi}(f) = \iint_{\sigma > 0} \frac{d\sigma}{\sigma} \ d\tau \ w(\sigma) \hat{\psi}_{\sigma, \tau}(f) \tilde{\chi}(\sigma, \tau).$$
$$(3)$$

The inverse Fourier transform of the left side is the **convolution operator** $\mathcal{W}$ ("stationary system") with *transfer function* $W(f)$. That is, let

$$w(t) \equiv \int_{-\infty}^{\infty} df \ e^{2\pi i f t} W(f).$$

Then $w(t)$ is the impulse response of $\mathcal{W}$ and

$$\mathcal{W}\chi(t) \equiv w * \chi(t) = \int_{-\infty}^{\infty} du \ w(t - u) \chi(u)$$
$$= \int_{-\infty}^{\infty} df \ e^{2\pi i f t} W(f) \hat{\chi}(f).$$

Taking the inverse Fourier transform of (3) gives

> $$w * \chi(t) = \int_{-\infty}^{\infty} df \ e^{2\pi i f t} W(f) \hat{\chi}(f)$$
> $$= \iint_{\sigma > 0} \frac{d\sigma}{\sigma} \ d\tau \ w(\sigma) \psi_{\sigma, \tau}(t) \tilde{\chi}(\sigma, \tau).$$

$\diamondsuit$

The meaning of $\diamondsuit$ is as follows:

We interpret $w(t)$ as the impulse response of a **filter**.

Then $\diamondsuit$ shows that the filtering can be done in the *scale domain* instead of the frequency domain, with

the correspondence

frequency filter $= W(f) \leftrightarrow w(\sigma) =$ scale filter
Fourier basis $= e^{2\pi i f t} \leftrightarrow \psi_{\sigma,\tau}(t) =$ wavelet basis
$$FT = \hat{\chi}(f) \leftrightarrow \tilde{\chi}(\sigma,\tau) = WT.$$

$\Diamond$ gives the reconstruction of $\chi(t)$ from $\tilde{\chi}(\sigma,\tau)$ as a special case: Choosing $w(\sigma) \equiv 1$ gives (for $f \neq 0$)

$$W(f) = \int_0^\infty \frac{d\sigma}{\sigma} \, S_\psi(f/\sigma) = \int_0^\infty \frac{d\xi}{\xi} \, S_\psi(\xi)$$
$$= \int_0^\infty \frac{d\xi}{\xi} \, |\hat{\psi}(\xi)|^2 \equiv C_\psi \,,$$

a *constant*. Hence $w * \chi(t) = C_\psi \chi(t)$, and $\Diamond$ gives

$$\chi(t) = C_\psi^{-1} \iint_{\sigma > 0} \frac{d\sigma}{\sigma} \, d\tau \, \psi_{\sigma,\tau}(t) \, \tilde{\chi}(\sigma,\tau). \qquad (\mathcal{R})$$

This is the *reconstruction formula* for the WT.

A *necessary* condition is

$$C_\psi \equiv \int_0^\infty \frac{df}{f} \, |\hat{\psi}(f)|^2 < \infty, \qquad (\mathcal{A})$$

called the **admissibility condition** for ψ.

Note that $\mathcal{A} \Rightarrow \hat{\psi}(0) = 0$, the *oscillation condition!*

$\mathcal{O}$ *must* be satisfied to make reconstruction possible, and $\psi(t)$ must therefore "wiggle." This is why $\psi(t)$ is called a "wavelet."

In $(\mathcal{R})$, all delays $\tau \in \mathbf{R}$ and all scales $\sigma > 0$ are used. But on physical grounds, we expect only $\tau > 0$ and $\sigma \approx 1$ to be relevant, since $|v| \ll c$ in radar. This is reflected in $(\mathcal{R})$ by the fact that for negative τ or large $|\log \sigma|$, the *coefficient function* $\tilde{\chi}(\sigma,\tau)$ is negligeably small since $\psi_{\sigma,\tau}(t)$ is a poor match for the actual return $\chi(t)$.

In fact, we can restrict the integral in $(\mathcal{R})$ to the values (σ,τ) expected to be most likely, simplifying

the computation. **Any other a priori knowledge about the target motion can be used similarly to "compress" $(\mathcal{R})$.**

But note that in **sonar**, the relative velocity of the target need not be negligeable compared to the propagation speed of the acoustic waves. (Consider a torpedo and a ship closing in at maximum speeds.) Then the reconstruction formula $(\mathcal{R})$ still works, with the proviso that larger scale factors become relevant. By contrast, the narrowband ambiguity function (discussed in Section 2 of this lecture) fails to take such situations into account.

Applications of $(\mathcal{R})$ and $\Diamond$:

1. Dense Target Environment: A single target with wideband parameters (σ,τ) gives a return $\psi_{\sigma,\tau}(t)$.

Therefore we *model* a dense distribution of targets (e.g., **clouds** or **dust in wind**) by a "target density function" $D(\sigma,\tau)$, with return

$$\chi(t) = \iint_{\sigma > 0} \frac{d\sigma}{\sigma} \, d\tau \, \psi_{\sigma,\tau}(t) \, D(\sigma,\tau).$$

The objective is to solve for $D(\sigma,\tau)$ in terms of χ. $(\mathcal{R})$ suggests that

$$D(\sigma,\tau) = C_\psi^{-1} \tilde{\chi}(\sigma,\tau),$$

but this is not quite right. A deeper analysis shows that $D(\sigma,\tau)$ cannot be found using a *single* wavelet ψ. Instead, a whole *class* of different basic wavelets (each with their family) must be used to estimate $D(\sigma,\tau)$. See [3, 5] and references therein.

2. Doppler Scale Processing: We can *process* the WT of the return, $\tilde{\chi}(\sigma,\tau)$, by choosing a weight function $w(\sigma)$ and using $\Diamond$. This might be done to remove noise or to concentrate on a particular range of scales, *refining* the constraint $\sigma \approx 1$.

Since σ is directly related to *velocity*, such **scale filtering** can be used for **moving target ID** and **clutter rejection.** *In the wideband case, this should work better than Doppler frequency filtering, since the Doppler frequency is frequency-dependent!*

LECTURE 3
Reduction to Time-Frequency Analysis
Sampling in Time-Scale versus Time-Frequency

1. The Narrowband Approximation

Under what conditions can *Doppler scaling* be replaced by a *Doppler frequency shift*? Again let ψ be a *bandpass filter:*

$$\hat{\psi}(f) \approx 0 \quad \text{unless} \quad 0 < f_1 \leq |f| \leq f_2$$

for some $0 < f_1 < f_2 < \infty$. Then positive and negative frequencies move in opposite directions. For example, suppose that $\sigma > 1$ (target closing in). Then

$$f > 0 \Rightarrow \sigma f > f$$
$$f < 0 \Rightarrow \sigma f < f.$$

Thus, in order to approximate scaling by shifts, we must eliminate the negative frequencies in $\hat{\psi}(f)$:

$$\hat{\psi}(f) \rightarrow \hat{\psi}^+(f) \equiv \begin{cases} \hat{\psi}(f) & \text{if } f > 0 \\ 0 & \text{if } f < 0. \end{cases}$$

The resulting time signal

$$\psi^+(t) = \int_0^\infty df \ e^{2\pi i f t} \hat{\psi}(f)$$

is *complex*. It is called the (Gabor) **analytic signal** of ψ because it extends analytically to the upper-half time plane. (See [3], Section 9.3 for a generalization to many variables, like space-time.)

All the frequency components in $\hat{\psi}^+(f)$ move in the same direction under scaling (to the *right* if $\sigma > 1$, to the *left* if $0 < \sigma < 1$.) But their motion still does not look like a uniform shift, unless the *endpoints* move by approximately the same amounts. To ensure this, we must assume that $\hat{\psi}^+(f)$ satisfies the following condition:

The *frequency-band* $[f_1, f_2]$ *is narrow*, in the sense that

$$\boxed{\frac{f_2 - f_1}{f_2 + f_1} \ll 1.} \qquad (\mathcal{N})$$

Then we can define the **carrier frequency** and the **bandwidth** of ψ as

$$f_c \equiv \frac{f_2 + f_1}{2} \quad \text{and} \quad b \equiv \frac{f_2 - f_1}{2} \ll f_c.$$

$\psi^+(t)$ can now be regarded as a slowly varying "message" imprinted on a rapidly oscillating carrier wave. In practice, $\psi^+(t)$ oscillates much too rapidly (at RF) to be sampled, hence the carrier is removed by *demodulation:*

$$\Psi(t) \equiv e^{-2\pi i f_c t} \psi^+(t).$$

This is the *complex envelope* or the **video signal** of ψ.

Its real and imaginary parts are called the **in-phase** and **quadrature components:**

$$I(t) \equiv \text{Re}\,\Psi(t) = \text{ in-phase component}$$
$$Q(t) \equiv \text{Im}\,\Psi(t) = \text{ quadrature component}.$$

A given frequency f in $[f_1, f_2]$ is shifted by

$$\Delta f \equiv \sigma f - f = (\sigma - 1)f = -\frac{2v}{c+v}f \approx -\frac{2v}{c}f,$$

since $|v| \ll c$. If the narrowband assumption $(\mathcal{N})$ holds, we can substitute f_c for f on the right to obtain

$$\Delta f \approx -\frac{2v}{c}f_c = -\frac{2v}{\lambda_c},$$

where $\lambda_c = c/f_c$ is the carrier wavelength.

The point is that *now the frequency shift*

$$\phi \equiv -\frac{2v}{\lambda_c} \qquad (1)$$

is independent of f.

In the narrowband approximation, all frequency components in $\hat{\psi}^+(f)$ undego a uniform **Doppler shift** ϕ given by (1).

The wideband parameters (σ, τ) are thus replaced by the **narrowband parameters** (ϕ, τ). These are related to the target parameters (r_0, v) by

$$\phi = -\frac{2v}{\lambda_c} \quad \text{and} \quad \tau = \frac{2r_0}{c},$$

where we have used $|v| \ll c$ in the second equation. The new objective is to estimate ϕ and τ, from which we obtain the range and range rate by

$$r_0 = \frac{c\tau}{2} \quad \text{and} \quad v = -\frac{\lambda_c \phi}{2}.$$

It can be shown [2] that in the narrowband approximation, the wavelets based on the analytic signal $\psi^+(t)$ become

$$\psi^+_{\sigma,\tau}(t) \rightarrow e^{2\pi f_c t}\, \Psi_{\phi,\tau}(t),$$

where

$$\Psi_{\phi,\tau}(t) \equiv e^{-2\pi i f_c \tau}\, e^{2\pi i \phi t}\, \Psi(t-\tau) \qquad (2)$$

is the video signal of $\psi_{\sigma,\tau}$. It is a *translated and modulated* version of $\Psi(t)$. The scaling has now been replaced by a modulation.

Equation (2) states that $\Psi_{\phi,\tau}(t)$ is a **phasor** rotating at the Doppler frequency ϕ. This explains why the *complex* video signals must be used. If only the in-phase or the quadrature components were used, then

the phasor would be replaced by a *harmonic oscillator* moving in a line, like $e^{2\pi i \phi t} \rightarrow \cos(2\pi\phi t)$. Although this is enough to determine $|\phi|$ and hence $|v|$, it does *not* determine the *sign* of v, hence the *direction* of the target motion (towards or away from the radar site). Clearly this is a very valuable piece of information!

Something similar to the mathematical transformations

$$\psi(t) \rightarrow \psi^+(t) \rightarrow \Psi(t) = e^{-2\pi i f_c t}\, \psi^+(t)$$

must actually be performed at the hardware level in order to "slow down" the signals so they can be sampled and processed. See Stimson [7] for a detailed explanation with wonderful illustrations.

The video signal of the measured return $\chi(t)$ is

$$X(t) \equiv e^{-2\pi i f_c t}\, \chi^+(t),$$

where $\chi^+(t)$ is the analytic signal of $\chi(t)$.

Claim: *In the narrowband approximation, the wideband ambiguity function becomes the windowed Fourier transform of $X(t)$ with respect to $\Psi(t)$ as the basic window.*

Recall that the wideband ambiguity function, expressed in the frequency domain, is

$$\tilde{\chi}(\sigma,\tau) \equiv \langle \psi_{\sigma,\tau}, \chi \rangle = \langle \hat{\psi}_{\sigma,\tau}, \hat{\chi} \rangle$$

$$= \frac{1}{\sqrt{\sigma}} \int_{-\infty}^{\infty} df\; e^{2\pi i f \tau/\sigma}\, \hat{\psi}(f/\sigma)^*\, \hat{\chi}(f)$$

$$= \tilde{\chi}^+(\sigma,\tau) + \tilde{\chi}^-(\sigma,\tau),$$

where

$$\tilde{\chi}^+(\sigma,\tau) = \frac{1}{\sqrt{\sigma}} \int_0^{\infty} df\; e^{2\pi i f \tau/\sigma}\, \hat{\psi}(f/\sigma)^*\, \hat{\chi}(f)$$

$$\tilde{\chi}^-(\sigma,\tau) \equiv \frac{1}{\sqrt{\sigma}} \int_{-\infty}^{0} df\; e^{2\pi i f \tau/\sigma}\, \hat{\psi}(f/\sigma)^*\, \hat{\chi}(f)$$

$$= \sqrt{\sigma} \int_0^{\infty} df\; e^{-2\pi i f \tau/\sigma}\, \hat{\psi}(-f/\sigma)^*\, \hat{\chi}(-f)$$

$$= \sqrt{\sigma} \int_0^{\infty} df\; e^{-2\pi i f \tau/\sigma}\, \hat{\psi}(f/\sigma)\, \hat{\chi}(f)^*$$

$$= \tilde{\chi}^+(\sigma,\tau)^*.$$

Thus

$$\tilde{\chi}(\sigma,\tau) = 2\mathrm{Re}\,\tilde{\chi}^+(\sigma,\tau).$$

But $\tilde{\chi}^+(\sigma,\tau)$ may be expressed directly in terms of the analytic signals $\psi^+_{\sigma,\tau}$ and χ^+ by

$$\tilde{\chi}^+(\sigma,\tau) = \langle \hat{\psi}^+_{\sigma,\tau}, \hat{\chi}^+ \rangle$$

$$= \langle \psi^+_{\sigma,\tau}, \chi^+ \rangle \rightarrow \langle \Psi_{\phi,\tau}, X \rangle.$$

where the last relation holds in the narrowband approximation. Note that the carrier phasor $e^{2\pi i f_c t}$, which is contained in both $\psi^+_{\sigma,\tau}(t)$ and $\chi^+(t)$, is *automatically* demodulated by matching the two analytic signals.

The **narrowband ambiguity function** is defined by matching the video signal $X(t)$ of the return with the video signal $\Psi_{\phi,\tau}(t)$ of the trial return:

$$\tilde{X}(\phi,\tau) \equiv \langle \Psi_{\phi,\tau}, X \rangle$$

$$= e^{2\pi i f_c \tau} \int_{-\infty}^{\infty} dt\; e^{-2\pi i \phi t}\, \Psi(t-\tau)^*\, X(t).$$

This is seen to be the **windowed Fourier transform** of $X(t)$, using $\Psi(t)$ as a basic window.

Thus we have shown that in the narrowband approximation, the ambiguity functions are related by

$$\tilde{\chi}(\sigma,\tau) \to 2\mathrm{Re}\ \tilde{X}(\phi,\tau).$$

Since the wideband ambiguity function is a prototype of time-scale (wavelet) analysis, we conclude:

> Time-frequency analysis may be regarded as the narrowband limit of time-scale analysis, where the basic window is the video signal of the basic wavelet.

$\tilde{X}(\phi,\tau)$ is used to determine (ϕ,τ) in the same way as $\tilde{\chi}(\sigma,\tau)$ is used to determine (σ,τ). Note first that

$$\|\Psi_{\phi,\tau}\|^2 = \|\Psi\|^2 = \|\psi^+\|^2 = \frac{1}{2}\ \|\psi\|^2 = \frac{1}{2}\ .$$

Hence Schwarz's inequality implies

$$|\tilde{X}(\phi,\tau)| \le \|\Psi_{\phi,\tau}\|\,\|X\| = \frac{1}{\sqrt{2}}\ \|X\|$$

$$|\tilde{X}(\phi,\tau)| = \frac{1}{\sqrt{2}}\ \|X\| \iff X(t) = C\Psi_{\phi,\tau}(t).$$

The narrowband parameters can therefore be estimated by maximizing $|\tilde{X}(\phi,\tau)|$. Alternatively, we can minimize the **narrowband error function**

$$\mathcal{E}(\phi,\tau) \equiv 1 - \frac{|\tilde{X}(\phi,\tau)|^2}{\|\Psi\|^2\,\|X\|^2}\ ,$$

since the Schwarz inequality states that

$$0 \le \mathcal{E}(\phi,\tau) \le 1 \quad \text{and}$$
$$\mathcal{E}(\phi,\tau) = 0 \iff X(t) = C\Psi_{\phi,\tau}(t).$$

Summary of Ambiguity Functional Methods

> The ambiguity functional $\tilde{\chi}(\beta)$ and error functional $\mathcal{E}(\beta)$ can be used to estimate an arbitrary *point-target* trajectory, with both the transmitting and receiving platforms in arbitrary and independent motions. When the radar is monostatic and the target motion is uniform in the line of sight, then $\tilde{\chi}(\beta)$ and $\mathcal{E}(\beta)$ reduce to the corresponding wideband functions $\tilde{\chi}(\sigma,\tau)$ and $\mathcal{E}(\sigma,\tau)$, which can be used to estimate the wideband target parameters (σ,τ). If, moreover, the outgoing signal $\psi(t)$ is narrowband and the target speed $|v| \ll c$, then the wideband functions further reduce to the narrowband functions $\tilde{X}(\phi,\tau)$ and $\mathcal{E}(\phi,\tau)$, which can be used to estimate the narrowband parameters (ϕ,τ). In this sense, *physical wavelet analysis is a generalization of ordinary (time-scale) wavelet analysis, and the latter is a generalization of time-frequency analysis. Radar provides the glue!*

2. Time-Scale *vs* Time-Frequency Sampling

Actual computations are numerical, hence use *sampling*. The wavelet transform naturally divides the frequency domain *logarithmcally* into proportional-width frequency bands, as in music, so that each band is sampled at a rate proportional to its mean frequency. In the narrowband approximation, the widths of all the frequency bands become equal. Hence in time-frequency analysis, all bands are sampled at the same rate. This *multirate* feature makes wavelet analysis much more efficient than time-frequency analysis. To show this, recall that for real $\psi(t)$ and $\chi(t)$,

$$\tilde{\chi}(\sigma,\tau) \doteq \sqrt{\sigma} \int_{-\infty}^{\infty} dt\ \psi(\sigma t - \tau)\chi(t)$$

$$= \frac{1}{\sqrt{\sigma}} \int_{-\infty}^{\infty} df\ e^{2\pi i f\tau/\sigma}\ \hat{\psi}(f/\sigma)^*\ \hat{\chi}(f)$$

$$= \sqrt{\sigma} \int_{-\infty}^{\infty} df\ e^{2\pi i f\tau}\ \hat{\psi}(f)^*\ \hat{\chi}(\sigma f)$$

$$= 2\mathrm{Re} \int_{0}^{\infty} df\ e^{2\pi i f\tau}\ \hat{\psi}(f)^*\ \hat{\chi}(\sigma f).$$

Suppose, for simplicity, that ψ is a bandpass filter with
$$\hat{\psi}(f) \approx 0 \quad \text{unless } 1 \le |f| \le 2.$$

Since $\psi(t)$ and $\chi(t)$ are real, it suffices to consider only positive frequencies, i.e., look at $\tilde{\chi}^+(\sigma,\tau)$. The above shows that $\tilde{\chi}^+(\sigma,\tau)$ depends on $\hat{\chi}(f)$ mainly in the band
$$\sigma \le f \le 2\sigma.$$

Therefore, to cover the *entire spectrum*, we sample $\tilde{\chi}^+(\sigma,\tau)$ at the scales
$$\sigma_m = 2^m, \quad m \in \mathbf{Z} \equiv \{0, \pm 1, \pm 2, \pm 3, \cdots\}.$$

That is,

$$\tilde{\chi}^+(2^m,\tau) \quad \text{covers the band} \quad B_m : 2^m \le f \le 2^{m+1}.$$

As m varies through the integers, all frequencies except DC ($f = 0$) are covered. Since the DC component does not propagate, this is OK.

Since the width of B_m is
$$2^{m+1} - 2^m = 2^m,$$

we sample $\tilde{\chi}^+(2^m,\tau)$ in τ at the Nyquist rate 2^m.

> *To recover all but the DC component of $\chi(t)$ from the wavelet transform $\tilde{\chi}(\sigma,\tau)$, sample $\tilde{\chi}^+(\sigma,\tau)$ at the discrete scale-time points $\sigma_m = 2^m$ and $t_{m,n} = n/2^m$, for all integers m, n.*

Note that the above sampling is *minimal*, and may in fact be too sparse for some wavelets. See [3], Chapter 6.

It *does* apply when ψ generates a *wavelet basis* defined by

$$\Psi_{m,n}(t) \equiv 2^{m/2}\psi(2^m t - n) = 2^{m/2}\psi(2^m(t - n/2^m)).$$

This includes all the compactly supported orthonormal wavelet bases discovered by Daubeshies [8] and their biorthogonal variants, as described in [9].

Just as a look at the *continuous* wavelet transform indicated how sampling must be done in the time-scale plane, so will a look at its narrowband limit, the windowed Fourier transform, reveal how to sample in the time-frequency plane.

Recall that the narrowband ambiguity function is
$$\tilde{X}(\phi,\tau) = \langle \Psi_{\phi,\tau}, X \rangle = \langle \hat{\Psi}_{\phi,\tau}, \hat{X} \rangle$$
$$= e^{2\pi i f_c \tau} \int_{-\infty}^{\infty} df \, e^{2\pi i f \tau} \hat{\Psi}(f)^* \hat{X}(f + \phi).$$

For simplicity, assume that the video signal Ψ is concentrated on the band $|f| \le 1/2$, i.e.,
$$\hat{\Psi}(f) \approx 0 \quad \text{unless } |f| \le \frac{1}{2}.$$

Then the above shows that $\tilde{X}(\phi,\tau)$ depends on $\hat{X}(f)$ mainly in the frequency band
$$\phi - \frac{1}{2} \le f \le \phi + \frac{1}{2}.$$

Thus, to cover the entire spectrum, we sample $\tilde{X}(\phi,\tau)$ at the Doppler frequencies
$$\phi_m = m, \quad m \in \mathbf{Z}.$$

That is,

$$\tilde{X}(\phi,\tau) \text{ covers the band } B'_m : m - \frac{1}{2} \le f \le m + \frac{1}{2}.$$

As m runs through the integers, the entire spectrum $\hat{X}(f)$ is covered. Since B'_m has unit width, the Nyquist rate for sampling $\tilde{X}(m,\tau)$ is also unity.

> *To recover $X(t)$ from its windowed Fourier transform $\tilde{X}(\phi,\tau)$, sample $\tilde{X}(\phi,\tau)$ at the discrete time-frequency points $\phi_m = m, \tau_n = n$, for all integers m, n.*

As in the case of time-scale analysis, this sampling is *minimal* and may actually be too sparse for some windows $\Psi(t)$. See [3], Chapter 5.

Note: The width of the *scaled* band B_m was 2^m, while the width of the *modulated* band B'_m is constant. This is what makes wavelet analysis so much more efficient because it allows high-frequency bands to be sampled at a proportionatey high rate.

Macroscopic Interpretations of (σ, τ) and (ϕ, τ)

Let $\psi(t)$ and $\Psi(t)$ be concentrated on $[-T, T]$.

Let $\hat{\psi}^+(f)$ be concentrated on $[f_c - b, f_c + b]$.

Let $\hat{\Psi}(f)$ be concentrated on $[-b, b]$.

In time-scale analysis, σ represents the *passband* of $\hat{\psi}_{\sigma,\tau}(f)$ and τ represents the *pass-time* of $\psi_{\sigma,\tau}(t)$. That is, they represent the *intervals*

$$\sigma \longleftrightarrow [\sigma(f_c - b), \sigma(f_c + b)]$$
$$\tau \longleftrightarrow [(\tau - T)/\sigma, (\tau + T)/\sigma].$$

Similarly, in time-frequency analysis, ϕ represents the passband of $\hat{\Psi}_{\phi,\tau}(f)$ and τ represents the pass-time of $\Psi_{\phi,\tau}(t)$. That is,

$$\phi \longleftrightarrow [\phi - b, \phi + b]$$
$$\tau \longleftrightarrow [\tau - T, \tau + T].$$

Thus, both sets (σ, τ) and (ϕ, τ) are actually **macrosopic frequency and time parameters**.

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Directivity: Radar with Pulsed-Beam Wavelets

1. Complex-Source Pulsed Beams

Given a small source centered at $\mathbf{x} \in \mathbf{R}^3$, responding to an impulse at time t. Let the response at $\mathbf{x}'$ at time t' be

$$K(\mathbf{x}', t' \,|\, \mathbf{x}, t) \equiv K(x' \,|\, x),$$

a solution of the wave equation in x' with a spatial source distribution representing the source at time t.

If the source is a *point* at $\mathbf{x}$, then

$$K(\mathbf{x}', t' \,|\, \mathbf{x}, t) = G(\mathbf{x}' - \mathbf{x}, t' - t) = G(x' - x),$$

where G is the *retarded Green function* for the wave equation:

$$G(\mathbf{x}, t) \equiv \frac{\delta(t - |\mathbf{x}|/c)}{4\pi|\mathbf{x}|}.$$

A simple model can be given for **extended** sources as follows:

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1. Find an *analytic extension* $K(x' \,|\, z)$ of $K(x' \,|\, x)$ *complex space-time source-locations:*

$$z = (\mathbf{z}, u) \in \mathbf{C}^4,$$

where

$$\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbf{C}^3 \quad \text{and} \quad u = t + is \in \mathbf{C}.$$

2. Interpret $K(x' \,|\, z)$ as a function of the *real* space-time observation point $x' = (\mathbf{x}', t') \in \mathbf{R}^4$.

Heyman and Felsen [10] showed that when the appropriate extension $K(x' \,|\, z)$ is chosen, it can be interpreted as a circular disk of radius $|\mathbf{y}|$, centered at $\mathbf{x}$ and radiating a pulsed beam in the direction of $\mathbf{y}$ at time t.

We will see that the imaginary space-time vector $(\mathbf{y}, s)$ must satisfy $|\mathbf{y}| \leq cs$ in order for the extension to be defined, and $|\mathbf{y}| < cs$ in order for it to be analytic. Heyman and Felsen usually choose the minimum value of s, i.e., $s = |\mathbf{y}|/c$. We will choose

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$s > |\mathbf{y}|/c$, so that $K(x' \,|\, z)$ is analytic. Since the value of s is related to the duration τ of the pulse by $\tau = s - |\mathbf{y}|/c$, leaving s free means that *all eight of the real variables $\mathbf{x}, \mathbf{y}, t, s$ have a physical interpretation in terms of the source.*

In [11] and [3], complementary extensions $K(z' \,|\, x)$ were derived and interpreted: *The location x of the source remained real, while the observation point became complex, i.e., $x' \to z' = x' + iy'$.*

It was shown that the eight real variables $x', y' \in \mathbf{R}^4$ can be interpreted as *phase-space coordinates*, something like an eight-dimensional time-frequency-space-wavenumber analysis of the waves. (See Chapter 4 of [11], and Chapters 9–11 of [3].)

The pulsed beams will be used to generalize the ambiguity functional formalism introduced in Lecture 1, which applied only for *point sources and targets*, to disks: The transmitting and receiving antennas will be *dishes*, and the target will be a circular disk. More complicated targets can then be composed from such disks.

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To derive the pulsed beams, recall $K(x' \mid x) = G(x' - x)$, where

$$G(x) \equiv G(\mathbf{x}, t) = \frac{\delta(t - |\mathbf{x}|/c)}{4\pi|\mathbf{x}|} .$$

We will continue $G(x)$ to $G(x + iy)$ by extending the numerator and the denominator separately.

1. Extension of $|\mathbf{x}|$:

$$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} \equiv \sqrt{\mathbf{x}^2}$$
$$\to \sqrt{\mathbf{z}^2} = \sqrt{\mathbf{x}^2 - \mathbf{y}^2 + 2i\mathbf{x} \cdot \mathbf{y}}.$$

A *branch* of the complex square root must be chosen.

2. Extension of $\delta(t - |\mathbf{x}|/c)$: First extend $\delta(t)$ to $\tilde{\delta}(t + is)$, then replace

$$\delta(t - |\mathbf{x}|/c) \to \tilde{\delta}(t + is - \sqrt{\mathbf{z} \cdot \mathbf{z}}/c).$$

The extension of $\delta(t)$ is defined by

$$\delta(t) = \int_{-\infty}^{\infty} df \; e^{2\pi i f t} \to \tilde{\delta}(t + is) \equiv \int_{0}^{\infty} e^{2\pi i f(t+is)},$$

which is the *analytic signal* of $\delta(t)$. For the integral to converge, we need $s > 0$. Then

$$\tilde{\delta}(t + is) = \frac{i}{2\pi} \frac{1}{t + is}, \quad s > 0.$$

Note that the boundary values of $\tilde{\delta}$ are related to δ by

$$\lim_{s \to 0^+} \left[2\mathrm{Re}\, \tilde{\delta}(t + is) \right] = \delta(t).$$

Thus

$$\tilde{G}(z) = \tilde{G}(\mathbf{z}, u) = \frac{\tilde{\delta}(t + is - \sqrt{\mathbf{z}^2}/c)}{(4\pi\sqrt{\mathbf{z}^2})}$$
$$= \frac{i}{8\pi^2(t + is - \sqrt{\mathbf{z}^2}/c)\sqrt{\mathbf{z}^2}} .$$

For $\tilde{\delta}(t + is - \sqrt{\mathbf{z}^2}/c)$ to be defined, we need

$$\mathrm{Im}\left(t + is - \sqrt{\mathbf{z}^2}/c\right) > 0, \Rightarrow \mathrm{Im}\,\sqrt{\mathbf{z}^2} < cs.$$

It can be shown that holds for all $\mathbf{x}$ if and only if

$$|\mathbf{y}| < cs.$$

An analytic extension of $G(x)$ is given by

$$\tilde{G}(z) = \tilde{G}(\mathbf{x} + i\mathbf{y}, t + is) = \frac{i}{8\pi^2(t + is - \sqrt{\mathbf{z}^2}/c)\sqrt{\mathbf{z}^2}},$$

with $|\mathbf{y}| < cs$.

This means that the real four-dimensional vector

$$y \equiv (\mathbf{y}, s) = \mathrm{Im}\, z$$

belongs to the *future cone*

$$V'_+ \equiv \{(\mathbf{y}, s) \in \mathbf{R}^4 : |\mathbf{y}| < cs\}.$$

Thus $\tilde{G}(x + iy)$ is analytic in the complex space-time domain

$$\mathcal{T}_+ \equiv \{x + iy \in \mathbf{C}^4 : y \in V'_+\},$$

called the **future tube** [3, 11].

For the above to make sense, we must choose a *branch* of $\sqrt{\mathbf{z}^2}$. In fact, the "branch points" are given by

$$\mathbf{z}^2 = \mathbf{x}^2 - \mathbf{y}^2 + 2i\mathbf{x} \cdot \mathbf{y} = 0,$$

or

$$\mathbf{x}^2 = \mathbf{y}^2 \quad \text{and} \quad \mathbf{x} \cdot \mathbf{y} = 0.$$

For fixed $\mathbf{y} \neq \mathbf{0}$, the first equation states that $\mathbf{x}$ belongs to the sphere of radius $|\mathbf{y}|$ centered at the origin, and the second equation further restricts $\mathbf{x}$ to the great circle in the plane orthogonal to $\mathbf{y}$.

Given $\mathbf{y} \neq \mathbf{0}$ and $s > |\mathbf{y}|/c$, the extended Green function $\tilde{G}(z)$ is therefore singular on the **branch circle**

$$\mathcal{C}_{\mathbf{y}} \equiv \{\mathbf{x} \in \mathbf{R}^3 : |\mathbf{x}| = |\mathbf{y}| \quad \text{and} \quad \mathbf{x} \cdot \mathbf{y} = 0\}.$$

Now fix $\mathbf{x}_0$ on $\mathcal{C}_{\mathbf{y}}$. Given any branch of $\sqrt{\mathbf{z}^2}$, follow this branch while moving around the circle of radius $R < |\mathbf{y}|$ centered at $\mathbf{x}_0$ and lying in the plane through the origin *orthogonal* to the plane of $\mathcal{C}_{\mathbf{y}}$. (The second circle is like a chain link around $\mathcal{C}_{\mathbf{y}}$!) Upon traversing this link, we find that $\sqrt{\mathbf{z}^2}$ has changed sign, hence we have moved to the *other* branch!

Thus, to define $\tilde{G}(z)$ uniquely, we must introduce a *branch sheet* $\mathcal{S}_\mathbf{y}$ (i.e., a two-dimensional analog of a branch cut), whose boundary is the branch circle:

$$\partial \mathcal{S}_\mathbf{y} = \mathcal{C}_\mathbf{y}.$$

$\tilde{G}(z)$ is discontinuous across $\mathcal{S}_\mathbf{y}$. Different choices for $\mathcal{S}_\mathbf{y}$ give different physical solutions [10]. Since we are interested in *compact* sources, we choose for $\mathcal{S}_\mathbf{y}$ the disk

$$\mathcal{S}_\mathbf{y} = \{\mathbf{x} \in \mathbf{R}^3 : |\mathbf{x}| \leq |\mathbf{y}| \quad \text{and} \quad \mathbf{x} \cdot \mathbf{y} = 0\},$$

which will be called the **source disk**. The reason for this name is as follows.

We define the *response* at the real space-time observation point $x' \in \mathbf{R}^4$ to an "impulse" disturbance at the *complex* space-time point $z = x + iy \in \mathcal{T}_+$ by

$$\boxed{K(x' \mid z) \equiv \tilde{G}(x' - z^*) = \tilde{G}(x' - x + iy).} \quad (1)$$

Note that the right-hand side is well-defined since $z \in \mathcal{T}_+$ implies that $x' - z^* \in \mathcal{T}_+$.

$K(x' \mid z)$ solves the *homogeneous* wave equation at all $x' = (\mathbf{x}', t') \in \mathbf{R}^4$ except when $\mathbf{x}' - \mathbf{x} \in \mathcal{S}_\mathbf{y}$, since K is discontinuous there. Recall that the EM version of $K(x' \mid z)$ is interpreted as a (Coulomb) potential, hence the EM fields are given by its derivatives with respect to x'. A discontinuity across $\mathcal{S}_\mathbf{y}$ therefore gives a delta-function singularity *on* $\mathcal{S}_\mathbf{y}$, which is interpreted as a source distribution concentrated on $\mathcal{S}_\mathbf{y}$.

The two sides of $\mathcal{S}_\mathbf{y}$ are not equivalent: The pulsed beam is radiated in the direction of $\mathbf{y}$ and not $-\mathbf{y}$. Thus we can interpret $K(x' \mid z)$ as the output of a dish antenna centered at $\mathbf{x}$ with *orientation* given by $\mathbf{y}$, emitting a pulse of duration (roughly) given by s at time t. The vector $\mathbf{y}$ therefore acts as a convenient "handle" by which the orientation of the antenna is controlled.

2. Pulsed-Beam Ambiguity Functionals

We now use pulsed beams to generalize the ambiguity functional formalism from point sources and point targets to disk antennas and disk targets.

Suppose that the transmitting antenna executes an arbitrary motion, including rotations and accelerations. This is parameterized by

$$z = \alpha(t) = x(t) + iy(t), \quad \text{where}$$
$$x(t) = (\mathbf{x}(t), t) \in \mathbf{R}^4 \quad \text{and} \quad y(t) = (\mathbf{y}(t), s) \in V'_+.$$

Assume $|\mathbf{y}(t)| = \text{constant} < cs$, but the direction of $\mathbf{y}(t)$ may vary to allow *rotations, tracking, scanning,* etc.

While executing this motion, the antenna is fed an input time signal $\psi(t)$. Then the output beam is

$$\Psi_\alpha(x') = \int_{-\infty}^\infty dt \, K(x' \mid \alpha(t)) \, \psi(t).$$

We call $\Psi_\alpha(x')$ the **extended physical wavelet** generated by $\psi(t)$ along $\alpha(t)$.

Given $\alpha(t)$, define the **emission operator** E_α by

$$\boxed{(E_\alpha \psi)(x') \equiv \int_{-\infty}^\infty dt \, K(x' \mid \alpha(t)) \, \psi(t).}$$

As before, E_α takes a function of one variable (the input signal) to a function of four variables (the output beam).

If the disk antenna is used as a *receiver*, it converts space-time waves into time signals. This defines a **reception operator** R_α. The simplest model for the received signal due to an incident wave $F(x')$ is, again,

$$\boxed{(R_\alpha F)(t) = g_\alpha F(\alpha(t)).}$$

where g_α is a gain factor. R_α measures the field along the complex trajectory $\dot{\alpha}(t)$. Again, more complicated receivers can be formulated which measure derivatives of F along $\alpha(t)$. (In EM, R_α could measure the *induced current* rather than the field.)

Since $\alpha(t)$ is complex, the value of $F(x')$ at $x' = \alpha(t)$ must be defined. We use the **analytic-signal transform** (AST), which extends F to complex space-time [3,11]:

$$\tilde{F}(x + iy) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} \, F(x + \tau y).$$

This is a space-time generalization of Gabor's analytic signals, reducing to the latter in one dimension (time signals).

If $F(x')$ is a solution of the homogeneous wave equation, then $\tilde{F}(x + iy)$ is analytic in the future tube $\mathcal{T}_+$ [3,11].

We model **reflection from a disk** as reception (by the *disk*, acting as a receiver) followed by re-emission:

$$F_{\text{refl}}(x') = (E_\alpha R_\alpha F)(x')$$
$$= g_\alpha \int_{-\infty}^{\infty} dt \, K(x' \,|\, \alpha(t)) \, \tilde{F}(\alpha(t)).$$

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Here g_α is re-interpreted as a **reflection coefficient**.

More complicated targets can be patched together from circular targets of various radii and orientations. The reflection coefficient then becomes a function defined over the target surface.

We now restate our elementary radar problem as follows:

A *disk transmitter* emits $\psi(t)$ while following $\alpha(t)$.

The wave is reflected from a *disk target* following $\beta_T(t)$.

The reflected wave is received by a *disk receiver* following $\gamma(t)$, giving a return $\chi(t)$.

All three trajectories $\alpha(t), \beta(t), \gamma(t)$ are *complex*, hence they include arbitrary scanning, tracking, and rotation as well as accelerations.

> **Problem:** Given the transmitted and received signals ψ, χ and the motions α, γ of the transmitter and receiver, estimate the target motion β_T.

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If we *knew* β_T, we could compute χ from

$$\chi = R_\gamma E_{\beta_T} R_{\beta_T} E_\alpha \psi.$$

Thus, to estimate β_T, we choose a *trial motion* $\beta(t)$ and compute the *trial return*, which we again denote by ψ_β:

$$\psi_\beta(t) = (R_\gamma E_\beta R_\beta E_\alpha \psi)(t)$$
$$= g_\gamma g_\beta \iint dt' \, dt'' \, K(\gamma(t) \,|\, \beta(t')) \, K(\beta(t') \,|\, \alpha(t'')) \, \psi(t'')$$

To estimate $\beta_T(t)$, we **match** ψ_β with the actual return $\chi(t)$ by taking the inner product of the two signals.

Again we denote the result by $\tilde{\chi}(\beta)$ and call it the **ambiguity functional** of χ (note that now $\psi_\beta(t)$ is complex):

$$\tilde{\chi}(\beta) \equiv \langle \psi_\beta, \chi \rangle \equiv \int_{-\infty}^{\infty} dt \, \psi_\beta(t)^* \chi(t).$$

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The Schwarz inequality implies

$$|\tilde{\chi}(\beta)| = |\langle \psi_\beta, \chi \rangle| \leq \|\psi_\beta\| \, \|\chi\|, \text{ and}$$
$$|\tilde{\chi}(\beta)| = \|\psi_\beta\| \, \|\chi\| \iff \chi(t) = C\psi_\beta(t).$$

Define the *normalized* ambiguity functional

$$\tilde{\chi}_N(\beta) \equiv \frac{\tilde{\chi}(\beta)}{\|\psi_\beta\|}.$$

Then

$$|\tilde{\chi}_N(\beta)| \leq \|\chi\| \text{ for all trial motions } \beta, \text{ and}$$
$$|\tilde{\chi}_N(\beta)| = \|\chi\| \iff \chi(t) = C\psi_\beta(t).$$

Thus, *to estimate* $\beta_T(t)$, *we must maximize* $\tilde{\chi}_N(\beta)$.

Equivalently, we can minimize the **error functional** defined by

$$\mathcal{E}(\beta) \equiv 1 - \frac{|\chi(\beta)|^2}{\|\psi_\beta\|^2 \, \|\chi\|^2},$$

since the Schwarz inequality states that

$$0 \leq \mathcal{E}(\beta) \leq 1 \quad \text{and} \quad \mathcal{E}(\beta) = 0 \iff \chi(t) = C\psi_\beta(t).$$

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$|\tilde{\chi}_N(\beta)|$ and $\mathcal{E}(\beta)$ attain their maximum and minimum values, respectively, only when the trial return is indistinguishable from the actual return.

Again, $\chi(t) = C\psi_\beta(t)$ does not guarantee $\beta(t) = \beta_T(t)$, since the return does not uniquely determine the target motion.

Reason: The functionals $\tilde{\chi}_N(b)$ and $\mathcal{E}(\beta)$ are not one-to-one, i.e., $\tilde{\chi}_N(\beta_1) = \tilde{\chi}_N(\beta_2)$ does not imply $\beta_1(t) = \beta_2(t)$.

Thus the radar problem is *inherently ambiguous*.

The best we can do is find *all* β with $|\tilde{\chi}_N(\beta)| = \|\chi\|$.

This can be called the **ambiguity class** of $\chi(t)$.

Waveform design problem: Find ψ so as to minimize the ambiguity classes.

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3. Multiple Targets and Multiple Reflections

We have assumed that there is only *one* target.

With N targets, we can approximate the return as

$$\psi_{\beta_1,\beta_2,\ldots,\beta_N} \approx \psi_{\beta_1} + \cdots + \psi_{\beta_N}.$$

But this ignores *multiple reflections*.

These can cause resonances (ringing) and must sometimes be taken into account.

For example, the signal received by the doubly-reflecting path $\alpha \to \beta_m \to \beta_n \to \gamma$ is

$$R_\gamma E_{\beta_n} R_{\beta_n} E_{\beta_m} R_{\beta_m} E_\alpha \psi,$$

which is a triple integral

Summing over such returns with $k = 0, 1, 2, \cdots$ reflections (where $k = 0$ is *no* reflection, applicable to bistatic radar) gives the complete N-target trial return $\psi_{\beta_1,\beta_2,\ldots,\beta_N}$.

To estimate $\beta_1, \beta_2, \ldots, \beta_N$, define the N-target ambiguity functional by matching $\psi_{\beta_1,\beta_2,\ldots,\beta_N}$ with χ:

$$\tilde{\chi}(\beta_1, \beta_2, \ldots, \beta_N) \equiv \langle \psi_{\beta_1,\beta_2,\ldots,\beta_N}, \chi \rangle.$$

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The Schwarz inequality may be used again to optimize the match, just as with a single target.

This method is reminiscent of **Feynman diagrams** [12], where fundamental processes are represented by multiple integrals with corresponding intuitive diagrams. Because the physics is built into the formalism from the beginning through the Green functions, our model can handle such complications in a conceptually straightforward (if computationally nontrivial) way. The resemblance to Feynman diagrams is no coincidence, and the present formalism may be modified to include quantum (photonic) aspects of radar simply by using **Feynman propagators** in place of the retarded Green functions.

4-18

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